

An equation of the form

$$Pdx + Qdy + Rdz = 0$$

where P, Q, R are functions of x, y, z

is called Total Differential equation or Single differential equation.

- Necessary and sufficient condition for integrability of total differential equation

$$Pdx + Qdy + Rdz = 0.$$

Necessary Condition:

Consider the total diff. eqn

$$Pdx + Qdy + Rdz = 0 \quad \dots \dots \dots \quad (1)$$

where P, Q, R are functions of x, y, z

Let (1) have an integral (solution)

$$u(x, y, z) = C \quad \dots \dots \dots \quad (2)$$

Then total differential du must be equal to $Pdx + Qdy + Rdz$ or to it multiplied by a function. But we know that

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \quad \dots \dots \quad (3)$$

Since, (2) is an integral of (1) and hence P, Q, R must be proportional to $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ respectively.

$$\therefore \frac{\frac{\partial u}{\partial x}}{P} = \frac{\frac{\partial u}{\partial y}}{Q} = \frac{\frac{\partial u}{\partial z}}{R} = \lambda(x, y, z) \text{ say.}$$

$$\therefore \frac{\partial u}{\partial x} = P\lambda, \frac{\partial u}{\partial y} = Q\lambda, \frac{\partial u}{\partial z} = R\lambda \quad \dots \dots \quad (4)$$

From the first two eqns of (4), we have

$$\frac{\partial}{\partial y} (P\lambda) = \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \cdot \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial x} (Q\lambda)$$

$$\text{or, } \lambda \frac{\partial P}{\partial y} + P \frac{\partial \lambda}{\partial y} = \lambda \frac{\partial Q}{\partial x} + Q \frac{\partial \lambda}{\partial x}$$

$$\text{or, } \lambda \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = Q \frac{\partial \lambda}{\partial x} - P \frac{\partial \lambda}{\partial y} \quad \dots \dots \dots (5)$$

Similarly,

$$\lambda \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) = R \frac{\partial \lambda}{\partial y} - Q \frac{\partial \lambda}{\partial z} \quad \dots \dots \dots (6)$$

$$\text{and } \lambda \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) = P \frac{\partial \lambda}{\partial z} - R \frac{\partial \lambda}{\partial x} \quad \dots \dots \dots (7)$$

Multiplying (5), (6) and (7) by R , P and Q resp'y and adding, we get

$$P \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) + Q \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) + R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = 0 \quad \dots \dots \dots (8)$$

This is, therefore, the necessary condition for integrability of the equation (1)

OR

$$\begin{vmatrix} P & Q & R \\ P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = 0$$

Sufficient Condition for integrability:

If the relation

$$P \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) + Q \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) + R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = 0 \quad \dots \dots \dots (1)$$

is satisfied, the equation has a primitive of the form $\phi = \text{constant}$.

Now, let the equation (1) be satisfied for the coefficients P, Q, R of the equation

$$Pdx + Qdy + Rdz = 0 \dots \dots \dots (2)$$

Then a similar relation holds for the coefficients of

$$\mu Pdx + \mu Qdy + \mu Rdz = 0 \dots \dots \dots (3)$$

where μ is any function of x, y, z .

Now, $Pdx + Qdy$ may be regarded as an exact differential. If it is not so, then multiplying the given equation by integrating factor $\mu(x, y, z)$, we can make it so.

$$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \dots \dots \dots (4)$$

and $\int(Pdx + Qdy) = V$

Then $\frac{\partial V}{\partial x} = P$ and $\frac{\partial V}{\partial y} = Q$

$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial z} \left(\frac{\partial V}{\partial x} \right) = \frac{\partial^2 V}{\partial z \cdot \partial x} \text{ and}$$

$$\frac{\partial Q}{\partial z} = \frac{\partial}{\partial z} \left(\frac{\partial V}{\partial y} \right) = \frac{\partial^2 V}{\partial z \cdot \partial y}$$

Substituting all these values in (1), we have

$$\frac{\partial V}{\partial x} \left(\frac{\partial^2 V}{\partial z \cdot \partial y} - \frac{\partial R}{\partial y} \right) + \frac{\partial V}{\partial y} \left(\frac{\partial R}{\partial x} - \frac{\partial^2 V}{\partial z \cdot \partial x} \right) = 0$$

$$\text{or, } \frac{\partial V}{\partial x} \cdot \frac{\partial}{\partial y} \left(\frac{\partial V}{\partial z} - R \right) - \frac{\partial V}{\partial y} \cdot \frac{\partial}{\partial x} \left(\frac{\partial V}{\partial z} - R \right) = 0$$

$$\text{or, } \begin{vmatrix} \frac{\partial V}{\partial x} & \frac{\partial}{\partial x} \left(\frac{\partial V}{\partial z} - R \right) \\ \frac{\partial V}{\partial y} & \frac{\partial}{\partial y} \left(\frac{\partial V}{\partial z} - R \right) \end{vmatrix} = 0$$

The equation shows that a relation independent of x and y exists between V and $(\frac{\partial V}{\partial z} - R)$.

Therefore $\frac{\partial V}{\partial z} - R$ can be expressed as a function of z and V alone. i.e. we can take

$$\left(\frac{\partial V}{\partial z}\right) - R = \phi(z, V)$$

$$\text{Now, } Pdx + Qdy + Rdz = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \left(\frac{\partial V}{\partial z} - \phi\right)dz$$

$$= \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz - \phi dz$$

$$= dV - \phi dz$$

Thus the equation $Pdx + Qdy + Rdz = 0$ may be written as $dV - \phi dz = 0$ which is an equation in two variables. Therefore, its integral may be taken of the form

$$f(z, V) = 0$$

Hence the condition is satisfied.

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